

The “Signature” of a Coherent System and Its Application to Comparisons Among Systems

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Abstract: Various methods and criteria for comparing coherent systems are discussed. Theoretical results are derived for comparing systems of a given order when components are assumed to have independent and identically distributed lifetimes. All comparisons rely on the representation of a system's lifetime distribution as a function of the system's "signature," that is, as a function of the vector $\mathbf{p} = (p_1, \dots, p_n)$, where p_i is the probability that the system fails upon the occurrence of the i th component failure. Sufficient conditions are provided for the lifetime of one system to be larger than that of another system in three different senses: stochastic ordering, hazard rate ordering, and likelihood ratio ordering. Further, a new preservation theorem for hazard rate ordering is established. In the final section, the notion of system signature is used to examine a recently published conjecture regarding componentwise and systemwise redundancy. © 1999 John Wiley & Sons, Inc. *Naval Research Logistics* 46: 507–523, 1999

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1. INTRODUCTION

An n -component system is said to be coherent if every component is relevant, i.e., has an effect on system performance, and if the system is monotone, i.e., the improvement of components cannot lead to a deterioration in system performance. The notion of coherence is central in reliability analysis since any system without it would rightly be judged to be fundamentally flawed and subject to alteration. While the performance characteristics of coherent systems have been the subject of considerable study, the development of comparisons among them is at present quite incomplete. In this paper, we propose a new approach based on a standardizing assumption that places the systems being compared on an equal footing. We will show that our approach provides a framework under which rather strong conclusions can be reached regarding the lifetime distributions of competing systems.

There are a variety of ways to compare two coherent systems. Comparison methods which induce a total ordering on the set of all coherent systems of a given size are typically based on

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comparing the values of a particular figure of merit for the systems (for example, the systems' expected lifetimes). Among the limitations of such approaches are the inevitable analytical difficulties in computing the desired figure of merit when a system is complex and the fact that any single numerical measure of a system's performance provides insight into just one aspect of the quality of that system. Methods which induce partial orderings among coherent systems tend to be based on structural comparisons which allow one to declare that a given system is better than another in some uniform sense. Typically, when two systems are ordered in this universal way, the corresponding ordering in terms of relevant figures of merit will hold as an easy consequence.

Let us consider in more detail some varied approaches to the comparison of two coherent systems of order n . Let $\mathbf{x} \in \{0, 1\}^n$ be the state vector of an n component system, where

$$x_i = \begin{cases} 1 & \text{if component } i \text{ works,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $i = 1, \dots, n$, and let $\phi_j(\mathbf{x})$ be the structure function for system j , that is, let

$$\phi_j(\mathbf{x}) = \begin{cases} 1 & \text{if components are in state } \mathbf{x} \text{ and the } j\text{th system works,} \\ 0 & \text{if components are in state } \mathbf{x} \text{ and the } j\text{th system fails,} \end{cases} \tag{2}$$

with $j = 1$ or 2 . Now suppose

$$\phi_1(\mathbf{x}) \leq \phi_2(\mathbf{x}) \quad \forall \mathbf{x} \in \{0, 1\}^n. \tag{3}$$

Then the second system will work under any condition under which system 1 works, so that system 2 is clearly a better system. For an interesting example of this mode of system comparison, see Block and Borges [2]. An example of two systems which are not comparable

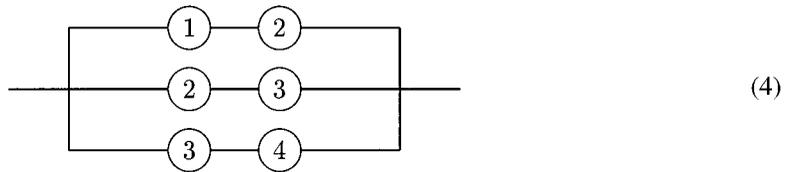


Figure 1. A consecutive 2-out-of-4 system.

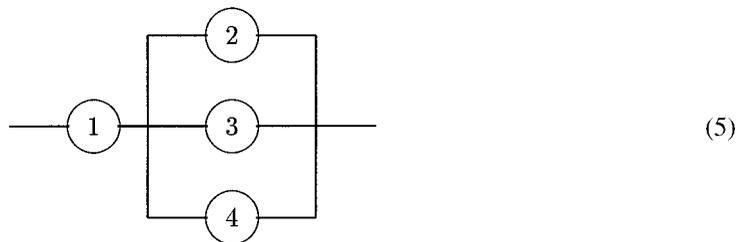


Figure 2. Two parallel systems in series.

relative to this ordering appears in Figures 1 and 2. For the state vector (0, 0, 1, 1), the system in (4) works and the system in (5) fails, while, for the state vector (1, 0, 0, 1), the system in (4) fails and the system in (5) works. We will return to the comparison of these two systems using the alternative method developed in Section 2.

To continue with our discussion of comparison methods, let us assume that two systems of order n have statistically independent components, and that the state X_i of each component is a Bernoulli random variable with parameter p_i . Then the reliability function $h_j(\mathbf{p})$ of system j is given by

$$h_j(\mathbf{p}) = P(\phi_j(\mathbf{X}) = 1).$$

If

$$h_1(\mathbf{p}) \leq h_2(\mathbf{p}) \quad \forall \mathbf{p} \in [0, 1]^n, \tag{6}$$

we would properly assert that system 2 is better than system 1. The fact that (3) implies (6) is obvious from the definition of the reliability function h . It is easy to show, in fact, that (3) and (6) are equivalent. Suppose, however, that we wish to compare the two systems when their components’ states are assumed to be identically distributed with common probability of success $p_i = p$. In that event, the reliability function depends on the single parameter p , and we would consider system 2 better than system 1 if

$$h_1(p) \leq h_2(p) \quad \forall p \in [0, 1]. \tag{7}$$

While (3) still implies (7), the converse is false. It can be shown, for example, that the system in (4) is better than the system in (5) in the sense of inequality (7) in spite of the fact that their structure functions are not comparable via the partial ordering induced by (3).

We should perhaps mention, in passing, a notable alternative approach to the comparison of systems. One might compare two systems, possibly of different orders n_1 and n_2 , with components of varying reliability $\mathbf{p}_1 = (p_{11}, \dots, p_{1n_1})$ and $\mathbf{p}_2 = (p_{21}, \dots, p_{2n_2})$. The first system would be judged better than the other if its reliability function $h_1(\mathbf{p}_1)$ is larger than $h_2(\mathbf{p}_2)$. Such an approach is taken by Proschan and Tsaturyan [7], who compare series-parallel (and parallel-series) systems for which component reliabilities differ in a well-defined way.

The comparisons above are cast in terms of the state of the system (as reflected by ϕ or h) at some fixed point in time. It is often of interest to compare system lifetime distributions, judging one system to be better than another if it ‘‘tends to last longer’’ in some specific sense. If the lifetimes of systems 1 and 2 are denoted by T_1 and T_2 , then one would certainly consider system 2 better than system 1 if

$$P(T_1 > t) \leq P(T_2 > t) \quad \forall t \geq 0, \tag{8}$$

that is, if T_1 is stochastically smaller than T_2 ($T_1 \leq^s T_2$). For a system ϕ of order n having statistically independent components whose lifetimes $\{T_i\}$ have distributions $\{F_i\}$, $i = 1, \dots, n$, the survival function $\bar{F}(t) \equiv P(T > t)$ of the system lifetime T may be written as

$$\bar{F}(t) = h_\phi[\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t)], \tag{9}$$

where $\bar{F}_i(t) \equiv 1 - F_i(t)$ for $i = 1, \dots, n$.

In Section 3, the comparison of systems is restricted to the case where component lifetimes are independent and identically distributed [as in the comparison in (7)]. While this would seem to be quite a restrictive assumption, we wish to put forward the argument that, in a sense, this is the most natural situation in which to compare systems, and that such a comparison constitutes the truest test of a preference for one system over another.

While there are many situations in which we might want to compare two systems, these situations tend to be characterized by the fact that the systems being compared are "similar." For example, we would probably not think of comparing the reliability functions of a television set and an automobile. We might, on the other hand, be very interested in comparing two automobiles. This suggests that comparison of systems makes the most sense when the systems being compared are of the same order and when the components of one system can be viewed as being comparable to those of the second system. Beyond the restriction to "similar systems," we would argue that the comparison is most meaningful when the component lifetimes of both systems are independent and identically distributed. We would certainly prefer to use a four-component series system each of whose components work with probability 0.9 than a four component parallel system whose components each work with probability 0.1. We would not, however, declare a series system to be better than a parallel system on that basis. When component lifetimes are taken as i.i.d., any remaining differences in system lifetime characteristics must be attributed to differences in the systems themselves. While we recognize that in particular applications, the lifetimes of the components actually employed might not reasonably be assumed to be i.i.d., the relative performance of systems under an i.i.d. assumption can still provide worthwhile information about system quality. We now turn to the development of our approach to comparing systems of order n with i.i.d. components.

2. THE SIGNATURE OF A COHERENT SYSTEM

A comprehensive treatment of system comparisons would have to account for all of the intricacies involved in a system's design and the effects of the varying distributions of component lifetimes and the possible dependencies among them. Comparisons are simplified substantially when component lifetimes are independent and have a common distribution F . Interestingly, the simplification goes beyond replacing a complex joint distribution of X_1, \dots, X_n by a simple function of the single distribution F . The i.i.d. assumption also allows us to focus on a particular, easily quantifiable aspect of a system's design which contains all the information that is relevant to the characteristics of system lifetime. In this section, we define the "signature" of a coherent system and illustrate its computation.

Consider a coherent system whose n components have lifetimes that are *i.i.d.* according to a continuous distribution F . Let X_1, \dots, X_n be the component lifetimes, and let T be the lifetime of the system. We note first that the system lifetime T has an order statistic equivalent, that is, that system failure always coincides with that of the i th component for some $i \in \{1, \dots, n\}$. Indeed, if $X_{(i)}$ represents the i th smallest component lifetime, $i = 1, \dots, n$, then we have $T \in \{X_{(1)}, X_{(2)}, \dots, X_{(n)}\}$ with probability one. Thus, we may identify the probability vector \mathbf{p} , where

$$p_i = P(T = X_{(i)}), \quad i = 1, \dots, n, \quad (10)$$

corresponding to each fixed coherent system of order n . It will shortly be shown that the lifetime distribution of a coherent system with i.i.d. components depends on the system's structural

design solely through the vector \mathbf{p} ; we will thus refer to \mathbf{p} as the system’s *signature*. In terms of the orderings of the component lifetimes X_1, X_2, \dots, X_n , one can define the signature \mathbf{p} as the probability vector with elements

$$p_i = \frac{\text{\# of orderings for which the } i\text{th failure causes system failure}}{n!}, \quad i = 1, \dots, n.$$

The vector \mathbf{p} is most easily obtained from the familiar cut set representation of system lifetime, namely,

$$T = \min_{1 \leq j \leq r} \max_{i \in K_j} X_i, \tag{11}$$

where K_1, \dots, K_r are the minimal cut sets of the system under study. The following example illustrates the computation of \mathbf{p} for a particular system.

EXAMPLE: Consider the system displayed in Figure 3 below.

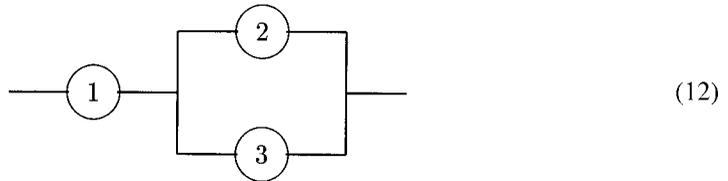


Figure 3. A series-parallel system in three components.

The minimal cut sets of this system are $K_1 = \{1\}$ and $K_2 = \{2, 3\}$. From (11), we may identify the system lifetime as

$$T = \min\{X_1, \max(X_2, X_3)\}.$$

The order statistic equivalent of T is shown below for each of the $3!$ orderings of the component lifetimes:

Ordering	\mathbf{T}
$X_1 < X_2 < X_3$	$X_{(1)}$
$X_1 < X_3 < X_2$	$X_{(1)}$
$X_2 < X_1 < X_3$	$X_{(2)}$
$X_2 < X_3 < X_1$	$X_{(2)}$
$X_3 < X_1 < X_2$	$X_{(2)}$
$X_3 < X_2 < X_1$	$X_{(2)}$

(13)

Since each ordering is equally likely, we may identify the signature of the system above as $\mathbf{p} = (1/3, 2/3, 0)$.

We now establish a fundamental property of a system's signature \mathbf{p} , namely, that the distribution of the system lifetime T , given i.i.d. components lifetimes with c.d.f. F , can be expressed as a function of \mathbf{p} and F alone.

THEOREM 1: Let X_1, \dots, X_n be the i.i.d. component lifetimes of a coherent system of order n , and let T be the system lifetime. Then

$$P(T > t) = \sum_{i=1}^n p_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}. \quad (14)$$

PROOF: Let π be a permutation of the positive integers $\{1, 2, \dots, n\}$ and let A_i be the collection of permutations for which $T = X_{(i)}$, that is, for which $T = X_{\pi_i}$, where $X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}$. Note that, given any permutation π , $X_{\pi_i} = X_{(i)}$ w.p. 1. We thus have

$$\begin{aligned} P(T > t) &= \sum_{i=1}^n P(T > t, \pi \in A_i) \\ &= \sum_{i=1}^n \sum_{\pi \in A_i} P(T > t, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}) \\ &= \sum_{i=1}^n \sum_{\pi \in A_i} P(X_{\pi_i} > t, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}). \\ &= \sum_{i=1}^n \sum_{\pi \in A_i} P(X_{(i)} > t, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}). \end{aligned} \quad (15)$$

But the events $\{X_{(i)} > t\}$ and $\{X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}\}$ in (15) are independent by Lemma 8.3.11 of Randles and Wolfe [8], since the former depends solely on the order statistic $X_{(i)}$ and the latter depends on the X 's only through the ranks of the original observations X_1, X_2, \dots, X_n . It thus follows that

$$\begin{aligned} P(T > t) &= \sum_{i=1}^n P(X_{(i)} > t) \sum_{\pi \in A_i} P(X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}) \\ &= \sum_{i=1}^n P(X_{(i)} > t) P(\pi \in A_i) \\ &= \sum_{i=1}^n p_i P(X_{(i)} > t) \\ &= \sum_{i=1}^n p_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}, \end{aligned}$$

completing the proof. \square

It is worth mentioning that the representation in (14) holds in contexts more general than the one studied here. In particular, it holds under the less stringent assumption that the component lifetimes X_1, \dots, X_n are exchangeable.

The representation in (14) is exploited in Samaniego [9] to obtain a useful representation of a system’s failure rate when F is absolutely continuous. In that paper, necessary and sufficient conditions are given for a coherent system in i.i.d. IFR components to itself have an IFR lifetime distribution. Our present interest is the *comparison* of two systems with i.i.d. components. As is clear from Eq. (14), the lifetime of a coherent system with i.i.d. components depends on the structure of the system only through the signature \mathbf{p} . Indeed, if two systems have the same signature, the stochastic behavior of their lifetimes is identical. It is natural to ask if two different coherent systems can have the same signature. The answer is yes; it is easily verified that the four-component system with minimal cut sets $\{1, 2\}, \{2, 4\}, \{3, 4\}$ has the same signature as the four-component system with minimal cut sets $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}$.

The discussion above shows that a certain amount of simplification is possible in characterizing the influence of system design on the distribution of system lifetime. The signature \mathbf{p} of a system serves as a compact but complete summary of the structure function ϕ , and also eliminates the duplication inherent in different structure functions whose impact on the distribution of system lifetime is identical. We develop below a further simplification, showing that the signature of a given system can be obtained without further computation from the signature of its dual.

Let ϕ be the structure function of a system of size n . The *dual* of ϕ is the structure function ϕ^D given by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}) \quad \forall \mathbf{x} \in [0,1]^n. \tag{16}$$

It follows that if \mathbf{y} is a cut vector of ϕ , i.e., $\phi(\mathbf{y}) = 0$, then $\phi^D(\mathbf{1} - \mathbf{y}) = \mathbf{1}$, that is, $\mathbf{1} - \mathbf{y}$ is a path vector of ϕ^D . Further, if A_1, \dots, A_k are the minimal path sets of ϕ , then A_1, \dots, A_k are the minimal cut sets of ϕ^D . Simple examples of duality include: the k -out-of- n system is the dual of the $(n - k + 1)$ -out-of- n system. For further discussion of duality, see Barlow and Proschan [1].

Since the number of coherent systems of order n can be large (growing exponentially in n), results which demonstrate relationships between particular systems serve to reduce the computational burden of obtaining the signatures of all systems of a given order. The following result cuts this burden in half.

THEOREM 2: Let \mathbf{p} be the signature of a fixed system ϕ whose n components have i.i.d. lifetimes, and let \mathbf{p}^D be the signature of its dual system ϕ^D . Then

$$p_i = p_{n-i+1}^D \quad \text{for } i = 1, 2, \dots, n. \tag{17}$$

PROOF: Given component lifetimes X_1, \dots, X_n , let T and T^D be the lifetimes of systems ϕ and ϕ^D , respectively. It suffices to show that

$$T = X_{(i)} \quad \text{if and only if} \quad T^D = X_{(n-i+1)}.$$

Employing the notation introduced in Theorem 1, let π represent a given permutation of the integers $\{1, 2, \dots, n\}$, and let $A_i =$ the set of permutations of $\{1, \dots, n\}$ such that $T = X_{\pi_i}$, where $X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}$; thus, $\pi \in A_i$ if and only if $T = X_{(i)}$. Now, assume that A_i is nonempty. For $\pi \in A_i$, let $\mathbf{x}_\pi \in \{0, 1\}^n$ be the state vector of the components at the time of system failure, that is, let \mathbf{x}_π be defined as

$$x_{\pi_j} = \begin{cases} 1 & \text{if } j > i, \\ 0 & \text{if } j \leq i. \end{cases}$$

Thus, \mathbf{x}_π has exactly i 0's and $(n - i)$ 1's. Moreover,

$$\phi(\mathbf{x}_\pi) = 0$$

and

$$\phi(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} > \mathbf{x}_\pi, \\ 0 & \text{if } \mathbf{y} \leq \mathbf{x}_\pi. \end{cases} \quad (18)$$

Now, by its definition, $\mathbf{1} - \mathbf{x}_\pi$ has $(n - i)$ 0's and i 1's. Further

$$\phi^D(\mathbf{1} - \mathbf{x}_\pi) = 1 - \phi(\mathbf{x}_\pi) = 1$$

and

$$\phi^D(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \geq \mathbf{1} - \mathbf{x}_\pi, \\ 0 & \text{if } \mathbf{y} < \mathbf{1} - \mathbf{x}_\pi. \end{cases} \quad (19)$$

From the latter characteristic of ϕ^D , we deduce that the $(n - i + 1)^{\text{st}}$ failure causes the failure of the dual system ϕ^D , i.e., $T^D = X_{\pi_{n-i+1}}$. Since this holds true for every $\pi \in A_i$, we have that $T^D = X_{(n-i+1)}$. It follows that if $p_i = P(T = X_{(i)})$ and $p_i^D = P(T^D = X_{(i)})$, then

$$(p_1, p_2, \dots, p_n) = (p_n^D, p_{n-1}^D, \dots, p_1^D), \quad (20)$$

that is, $p_i = p_{n-i+1}^D$ for $i = 1, 2, \dots, n$, as claimed. \square

We now proceed to our investigation of comparisons among coherent systems of a given order based on the properties of the systems' signatures.

3. COMPARING SYSTEM LIFETIMES

In this section, we develop three different scenarios for comparing the performance of coherent systems. Our three results feature increasingly stringent requirements on system signatures and demonstrate that these lead to correspondingly stronger conclusions regarding the distributions of system lifetime. We begin with a result which examines the consequence of the stochastic ordering of two signatures. For two discrete distributions \mathbf{p}_1 and \mathbf{p}_2 on the integers $\{1, \dots, n\}$, we write $\mathbf{p}_1 \leq^{\text{st}} \mathbf{p}_2$ if and only if

$$\sum_{i=j}^n p_{1i} \leq \sum_{i=j}^n p_{2i} \quad \text{for } j = 1, 2, \dots, n. \tag{21}$$

It is obvious that ordered structure functions, i.e., $\phi_1(\mathbf{x}) \leq \phi_2(\mathbf{x})$ as in (3), will imply ordered signatures, i.e., $\mathbf{p}_1 \leq^{st} \mathbf{p}_2$. The systems displayed in (4) and (5) constitute an example which shows that the converse implication fails. The relation between stochastically ordered signatures and system lifetimes is as follows:

THEOREM 3: Let \mathbf{p}_1 and \mathbf{p}_2 be the signatures of the two systems of order n , and let T_1 and T_2 be their respective lifetimes. If $\mathbf{p}_1 \leq^{st} \mathbf{p}_2$, then $T_1 \leq^{st} T_2$.

PROOF: We may rewrite the representation in (14) in a more convenient form:

$$\begin{aligned} P(T > t) &= \sum_{i=1}^n p_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (1 - F(t))^{n-j} \\ &= \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^n p_i \right) \binom{n}{j} (F(t))^j (1 - F(t))^{n-j}. \end{aligned}$$

We thus have, by virtue of the assumption that $\mathbf{p}_1 \leq^{st} \mathbf{p}_2$, that

$$\begin{aligned} P(T_1 > t) &= \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^n p_{1i} \right) \binom{n}{j} (F(t))^j (1 - F(t))^{n-j} \\ &\leq \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^n p_{2i} \right) \binom{n}{j} (F(t))^j (1 - F(t))^{n-j} \\ &= P(T_2 > t) \quad \forall t \geq 0, \end{aligned}$$

which is equivalent to $T_1 \leq^{st} T_2$. \square

It is easy to verify that the five different systems of order 3 are totally ordered in the sense of the theorem above. The 20 different coherent systems of order 4 cannot be totally ordered in this way. For example, the systems in Figure 4 below are not comparable by the method developed in this paper. While short of providing a total ordering, Theorem 2 sheds considerable light on the relative merits of various systems of order 4. Of the 190 possible pairwise comparisons among the systems of order 4, Theorem 3 may be applied to 180 pairs, identifying in each of these cases the system whose lifetime is stochastically larger when component lifetimes are i.i.d. Returning to the comparison of the two systems of order 4 displayed in (4) and (5), it can easily be verified that the signature of the system in (4) is $\mathbf{p} = (0, 1/2, 1/2, 0)$, while the signature of the system in (5) is $(1/4, 1/4, 1/2, 0)$. It follows from Theorem 2 that, given i.i.d. components, the system in (5) has a stochastically smaller lifetime.

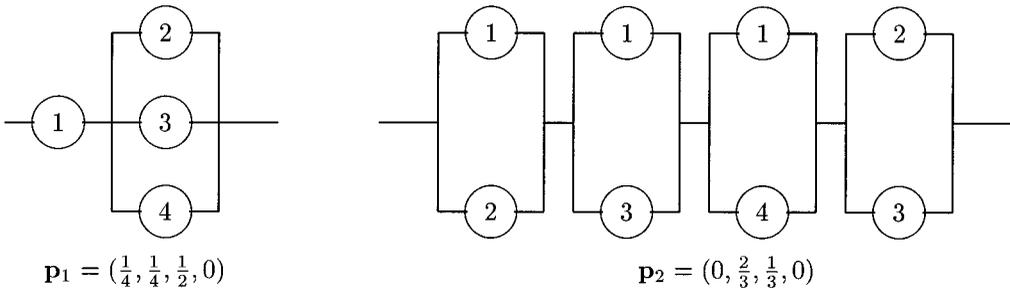


Figure 4. Noncomparable systems of order 4.

We now examine the implications of a stronger form of ordering between two system signatures. Let X_1 and X_2 be random variables with survival functions \bar{F}_1 and \bar{F}_2 , respectively. Then $X_1 \leq^{hr} X_2$ (i.e., X_1 is smaller than X_2 in the hazard rate ordering) if and only if the ratio

$$\frac{\bar{F}_2(x)}{\bar{F}_1(x)}$$

is nondecreasing for $x < F_1^{-1}(1)$. For two discrete distributions \mathbf{p}_1 and \mathbf{p}_2 on the set $\{1, \dots, n\}$, we say that $\mathbf{p}_1 \leq^{hr} \mathbf{p}_2$ if and only if

$$\frac{\sum_{j=i}^n p_{2j}}{\sum_{j=i}^n p_{1j}}$$

is nondecreasing in i . The following lemma is proved in Joag-dev, Kochar, and Proschan [5]:

LEMMA 1: Let α and β be real valued functions such that β is nonnegative and α/β and β are nondecreasing. If $X_i \sim F_i$, $i = 1, 2$, and $X_1 \leq^{hr} X_2$, then

$$\frac{\int_{-\infty}^{\infty} \alpha(x) dF_1(x)}{\int_{-\infty}^{\infty} \beta(x) dF_1(x)} \leq \frac{\int_{-\infty}^{\infty} \alpha(x) dF_2(x)}{\int_{-\infty}^{\infty} \beta(x) dF_2(x)}. \tag{22}$$

THEOREM 4: Let \mathbf{p}_1 and \mathbf{p}_2 be the signatures of two coherent systems, and let \mathbf{T}_1 and \mathbf{T}_2 be the corresponding system lifetimes. If $\mathbf{p}_1 \leq^{hr} \mathbf{p}_2$ then $T_1 \leq^{hr} T_2$.

PROOF: The survival function of T_1 may be written as

$$P(T_1 > t) = \sum_i p_{1i}P(X_{(i)} > t).$$

We must prove that

$$\frac{\sum_i p_{2i}P(X_{(i)} > t_2)}{\sum_i p_{1i}P(X_{(i)} > t_2)} \geq \frac{\sum_i p_{2i}P(X_{(i)} > t_1)}{\sum_i p_{1i}P(X_{(i)} > t_1)} \quad \forall t_1 < t_2, \tag{23}$$

or equivalently

$$\frac{\sum_i p_{1i}P(X_{(i)} > t_2)}{\sum_i p_{1i}P(X_{(i)} > t_1)} \leq \frac{\sum_i p_{2i}P(X_{(i)} > t_2)}{\sum_i p_{2i}P(X_{(i)} > t_1)} \quad \forall t_1 < t_2.$$

To employ Lemma 1, we make the identification $\alpha(i) = P(X_{(i)} > t_2)$ and $\beta(i) = P(X_{(i)} > t_1)$, and identify F_j as the discrete distribution \mathbf{p}_j for $j = 1, 2$. First, we note that

$$\frac{\alpha(i)}{\beta(i)} = \frac{P(X_{(i)} > t_2)}{P(X_{(i)} > t_1)} \tag{24}$$

is an increasing function of i . The monotonicity of the ratio in (24) follows from the fact that

$$\begin{aligned} \frac{P(X_{(i+1)} > t_2)}{P(X_{(i+1)} > t_1)} &\geq \frac{P(X_{(i)} > t_2)}{P(X_{(i)} > t_1)} \quad \forall t_1 \leq t_2 \\ \Leftrightarrow \frac{P(X_{(i+1)} > t_2)}{P(X_{(i)} > t_2)} &\geq \frac{P(X_{(i+1)} > t_1)}{P(X_{(i)} > t_1)} \quad \forall t_1 \leq t_2 \\ \Leftrightarrow \frac{\bar{F}_{(i+1)}(t)}{\bar{F}_{(i)}(t)} &\text{ is an increasing function of } t. \end{aligned}$$

The latter fact follows since $X_{(i+1)} \geq^{hr} X_{(i)}$, a result which holds as long as the X_i ’s are independent, whether they are identically distributed or not (see, for example, Boland, El-Newehi, and Proschan [4]).

Secondly, we note that $\beta(i) = P[X_{(i)} > t]$ is increasing in i since the successive order statistics are stochastically ordered. The required result, i.e., inequality (23), thus follows from (22) under the assumption that $\mathbf{p}_1 \leq^{hr} \mathbf{p}_2$. \square

Our next result studies the effect of likelihood ratio ordering between system signatures. Let X_1 and X_2 be real valued random variables with respective densities f_1 and f_2 . Then $X_1 \leq^{lr} X_2$ (i.e., X_1 is smaller than X_2 in the likelihood ratio ordering) if and only if the ratio

$$\frac{f_2(x)}{f_1(x)}$$

is nondecreasing in x . For two discrete distributions \mathbf{p}_1 and \mathbf{p}_2 on $\{1, 2, \dots, n\}$, $\mathbf{p}_1 \leq^{lr} \mathbf{p}_2$ if and only if p_{2i}/p_{1i} is nondecreasing in i .

THEOREM 5: Let T_1 and T_2 denote the lifetimes of two coherent systems in i.i.d. components with signatures \mathbf{p}_1 and \mathbf{p}_2 , respectively. If $\mathbf{p}_1 \leq^{lr} \mathbf{p}_2$, then $T_1 \leq^{lr} T_2$.

PROOF: For $j = 1, 2$, the survival function of T_j is

$$\bar{F}_j(t) = \sum_{i=1}^n p_{ji} P(X_{(i)} > t)$$

and the corresponding probability density function is

$$f_j(t) = \sum_{i=1}^n p_{ji} f_{(i)}(t).$$

It is sufficient to prove that for any real number c , the function

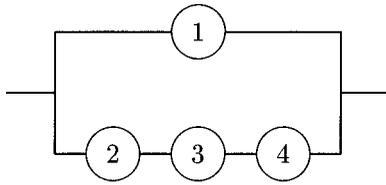
$$\begin{aligned} g(t) &= \sum_{i=1}^n p_{2i} f_{(i)}(t) - c \sum_{i=1}^n p_{1i} f_{(i)}(t) \\ &= \sum_{i=1}^n [p_{2i} - cp_{1i}] f_{(i)}(t) \end{aligned}$$

has at most one change of sign from negative to positive as t goes from 0 to ∞ . Since $\mathbf{p}_1 \leq^{lr} \mathbf{p}_2$, p_{2i}/p_{1i} is nondecreasing in i and, as a result, the sequence $\{p_{2i} - cp_{1i}\}$ has at most one change of sign from negative to positive as i goes from 1 to n . Since in the i.i.d. case, $X_{(i-1)} <^{lr} X_{(i)}$ for any i , we have that $f_{(i)}(t)/f_{(i-1)}(t)$ is nondecreasing in t . That is, the function $f_{(i)}(t)$ is Totally Positive of Order 2 (TP_2) in (i, t) . It follows from the variation diminishing property of TP_2 functions (see Karlin [6]) that $g(t)$ has at most one change of sign from negative to positive as t increases from $-\infty$ to ∞ . This completes the proof. \square

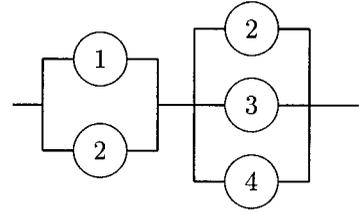
The results above show that the precise characteristics of a coherent system's signature have a direct effect on its lifetime distribution. As is well known (see Shaked and Shanthikumar [10]), the orderings we have discussed are increasingly stringent, with $\mathbf{p}_1 \leq^{lr} \mathbf{p}_2 \Rightarrow \mathbf{p}_1 \leq^{hr} \mathbf{p}_2 \Rightarrow \mathbf{p}_1 \leq^{st} \mathbf{p}_2$. Examples of systems with signatures satisfying some but not all these order relations are displayed in Figures 5 and 6.

There are a variety of other implications one may draw about the lifetime distribution of a system based on the representation (14) of that distribution as a function of system signatures. For example, it was shown in Samaniego [9] that a coherent system with n i.i.d. IFR components was IFR if and only if the rational function ψ , given by

$$\psi(u) = \frac{\sum_{i=0}^{n-1} (n-i)p_{i+1} \binom{n}{i} u^i}{\sum_{i=0}^{n-1} (\sum_{j=i+1}^n p_j) \binom{n}{i} u^i}, \tag{25}$$

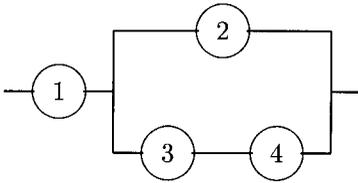


$$\mathbf{p}_1 = (0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$$

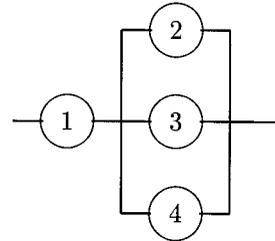


$$\mathbf{p}_2 = (0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$$

Figure 5. Systems for which $\mathbf{p}_1 \leq^{st} \mathbf{p}_2, \mathbf{p}_1 \not\leq^{hr} \mathbf{p}_2, \mathbf{p}_1 \not\leq^{lr} \mathbf{p}_2$.



$$\mathbf{p}_1 = (\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$$



$$\mathbf{p}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$$

Figure 6. Systems for which $\mathbf{p}_1 \leq^{st} \mathbf{p}_2, \mathbf{p}_1 \leq^{hr} \mathbf{p}_2, \mathbf{p}_1 \not\leq^{lr} \mathbf{p}_2$.

is increasing for $u \in (0, \infty)$. Using the failure rate representation in that paper, namely,

$$r_T(x) = \psi\left(\frac{F(t)}{F(t)}\right) \cdot r(t), \tag{26}$$

which is valid when F is absolutely continuous with failure rate r , one may easily establish the following ordering result.

THEOREM 6: Let ϕ be a fixed system in i.i.d. components, and let F_1, F_2 be two absolutely continuous lifetime distributions whose failure rates satisfy

$$r_1(x) \leq r_2(x) \quad \forall x.$$

For $i = 1, 2$, let T_i be the lifetime of the system ϕ when its components have i.i.d. lifetimes drawn from F_i , and suppose that the function ψ in (25) is increasing on $(0, \infty)$. Then

$$r_{T_1}(x) \leq r_{T_2}(x) \quad \forall x,$$

that is, hazard rate ordering of component lifetimes implies hazard rate ordering of system lifetimes.

4. AN APPLICATION INVOLVING REDUNDANCY

As a final example demonstrating the utility of system signatures in the comparison of particular systems, we consider a problem posed in the recent literature regarding the comparative performance of system vs. component redundancy. It is, of course, well known that componentwise redundancy is more effective than system-wise redundancy. Theorem 2.4 in Barlow and Proschan [1] states this domination in terms of the structure functions of the two designs ($\phi_1 \leq \phi_2$). Assuming i.i.d. component lifetimes, this then implies domination in terms of signatures ($\mathbf{p}_1 \leq^{st} \mathbf{p}_2$) and thus in terms of system lifetimes ($T_1 \leq^{st} T_2$). But other questions of interest arise in this context. For example, in this same i.i.d. setting, is it possible to conclude that $T_1 \leq^{hr} T_2$? Boland and El-Newehi [3] showed that this latter implication did not hold in general, but conjectured that hazard rate ordering of system lifetimes does hold when the two types of redundancy are applied to a k -out-of- n system in i.i.d. components.

The conjecture above has now been proven true; indeed, Singh and Singh [11] establish the stronger implication $T_1 \leq^{lr} T_2$ for componentwise over systemwise redundancy of k -out-of- n systems. However, their proof involves a technical lemma establishing several delicate inequalities, and requires the tacit assumption that the underlying component lifetime distribution F is absolutely continuous. We provide an alternative proof here that is both simpler (i.e., based on elementary combinatorics) and more general than that given by Singh and Singh [11]. Our proofs of the following two theorems hold for arbitrary continuous distributions F . A direct comparison of system signatures, together with an application of our Theorem 5, then proves the conjecture.

A 2 out of 3 system with redundancy at the system level is pictured in Figure 7, while a 2-out-of-3 system with componentwise redundancy is pictured in Figure 8.

We first establish the general form of the signatures of these two types of systems.

THEOREM 7: The signature \mathbf{p} of a k -out-of- n system with systemwise redundancy is

$$p_{2n-2k+2+r} = \frac{\binom{n-1}{k-1} \binom{n}{k-1-r}}{\binom{2n-1}{2k-2-r}} \quad \text{for } r = 0, 1, \dots, k-1 \tag{27}$$

with $p_i = 0$ for $1 \leq i < 2n - 2k + 2$ and for $2n - k + 1 < i \leq 2n$.

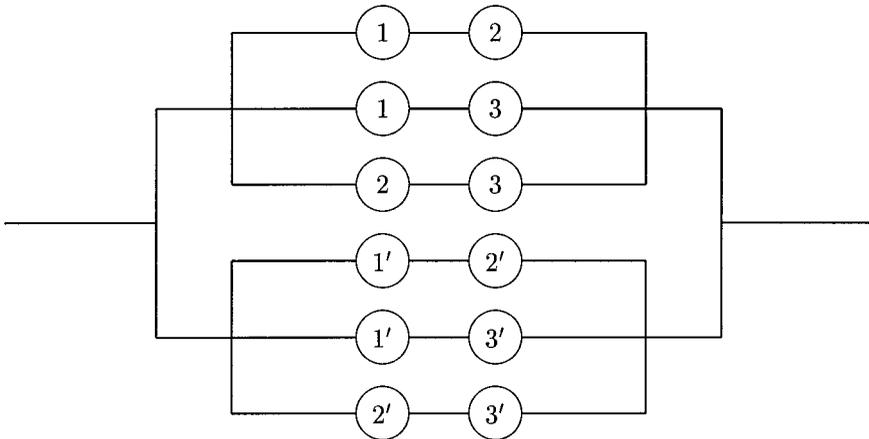


Figure 7. Systemwise redundancy.

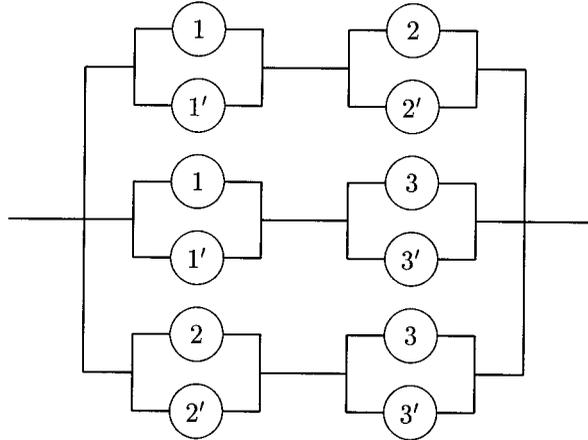


Figure 8. Componentwise redundancy.

PROOF: For $r = 0, \dots, k - 1$, we give a combinatorial proof which accounts for a ratio composed of: (i) the number of orderings of the $2n$ component failures for which the system fails upon the $(2n - 2k + 2 + r)$ th failure to (ii) the number of all possible orderings [that is, $(2n)!$] of component failures. Since system failure is only possible when at least $(n - k + 1)$ failures occur among the original and among the back up components, it is clear that $p_i = 0$ if $i < 2n - k + 2$. Further, the maximal number of possible failures that can occur without causing failure to the system is $2n - k$ (e.g., $n - k$ originals and n backups). It is thus clear that $p_i = 0$ for $i > 2n - k + 1$.

Now, consider orderings for which the $(2n - 2k + 2 + r)$ th component failure is fatal to the system. The number of such orderings is the product of the following factors: there are **2** ways to select batch #1, the batch of components (original or back up) from which the component failure fatal to the system will be drawn; there are $\binom{n - k + 1}{n - k + 1}$ ways to select the $(n - k + 1)$ components that fail among the n in batch #1; there are $\binom{n - k + 1 + r}{n - k + 1 + r}$ ways to select the $(n - k + 1 + r)$ components that fail prior to system failure among the n in batch #2, there are $(n - k + 1)$ ways to select the failed component which is fatal to the system; there are $(2n - 2k + r + 1)!$ orderings of the $(2n - 2k + r + 1)$ component failures that occur prior to the component failure fatal to the system; and there are $(2k - r - 2)!$ orderings of the remaining (hypothetical) component failures following system failure. It is easy to show that the product of these terms, divided by $(2n)!$, reduces to the expression for $p_{2n - 2k + 2 + r}$ in (27). \square

THEOREM 8: The signature of a k -out-of- n system with redundancy at the component level is given by

$$p_{2n - 2k + 2 + r} = \frac{\binom{n - 1}{k - 1} \binom{k - 1}{r} 2^r}{\binom{2n - 1}{2k - 2 - r}} \quad \text{for } r = 0, 1, \dots, k - 1, \tag{28}$$

with $p_i = 0$ for $1 \leq i < 2n - 2k + 2$ and for $2n - k + 1 < i \leq 2n$.

PROOF: As Figure 8 suggests, it is helpful to think of the system of interest here as a k -out-of- n system in n modules, each module being a small parallel system. For such a system to fail, we must have precisely $(n - k + 1)$ failed modules. For reasons similar to those cited in the proof of Theorem 7, we have $p_i > 0$ if and only if $2n - 2k + 2 \leq i \leq 2n - k + 1$.

Now, consider orderings of component failures for which the component fatal to the system is the $(2n - 2k + 2 + r)$ th component to fail. The number of such orderings is the product of the following factors: there are $\binom{n-k+1}{n-k+1}$ ways to select the $(n - k + 1)$ modules whose failure results in system failure; there are $\binom{k-r-1}{r}$ ways to select r other modules in which single component failures have occurred prior to system failure; there are 2^r ways to select a single failed component from the component pair in each of the r modules containing exactly one failure; there are $(2n - 2k + 2)$ ways to select the component whose failure is fatal to the system from among the $2n - 2k + 2$ components in the first $n - k + 1$ modules to fail; there are $(2n - 2k + 1 + r)!$ orderings of the $(2n - 2k + 1 + r)$ components that fail before the component whose failure is fatal to the system; and there are $(2k - 2 - r)!$ orderings of the remaining (hypothetical) component failures following system failure. The product of the terms above, divided by $(2n)!$, can easily be reduced to the expression for $p_{2n-2k+2+r}$ in (28). \square

Armed with the signature formulae for k -out-of- n systems under systemwise and componentwise redundancy, we are now in a position to establish the desired result.

THEOREM 9: For $1 \leq k \leq n$, let T_1 be the lifetime of a k -out-of- n system with i.i.d. components under systemwise redundancy, and let T_2 be the corresponding lifetime of the system under componentwise redundancy. Then $T_1 \leq^{lr} T_2$.

PROOF: For $r = 0, 1, \dots, k - 1$, we have from Theorems 7 and 8 that

$$s(r) \equiv \frac{p_{2n-2k+2+r}^{(2)}}{p_{2n-2k+2+r}^{(1)}} = \frac{\binom{k-1}{r} 2^r}{\binom{n}{k-r-1}}.$$

Now $s(r)$ is proportional to the function

$$t(r) = \frac{(n - k + r + 1)! 2^r}{r!}.$$

The fact that $s(r)$ is nondecreasing in r is implied by the fact that the inequality $t(r+1)/t(r) \geq 1$ holds for all $r \geq 0$. Given the monotonicity of s , it follows from Theorem 5 that $T_1 \leq^{lr} T_2$. \square

Note that the fact that $p^{(1)} \leq^{lr} p^{(2)}$, as shown in Theorem 8, implies (by virtue of Theorem 2) that the signatures of the two systems that are the duals of those in Theorem 8 satisfy the opposite inequality, that is, $p_D^{(1)} \leq^{lr} p_D^{(2)}$. This in turn implies that the lifetimes of the dual systems satisfy $T_1^D \leq^{lr} T_2^D$. This result is an immediate consequence of the approach taken here, but is not easily derived as a consequence of the developments in Singh and Singh [11].

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